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Bend strength of alumina ceramics: A comparison of Weibull statistics with other statistics based on very large experimental data set

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Abstract

We have performed a statistical evaluation of 5100 experimental values of the bend strength of test pieces from a serial production of alumina products. The Weibull distribution was compared to three other, commonly known, 2-parametric distributions in order to reveal which of them best matches the experiments. The maximum-likelihood method was used to evaluate the corresponding parameters, and then a *Q*–*Q* plot was used for all the statistics. We confirmed that the Weibull distribution describes the experimental strengths most accurately.

Keywords: Bend strength; Weibull statistics; Maximum-likelihood method; *Q*–*Q* plot

1. Introduction

The scatter in the values of strength measured in typical mechanical tests for brittle materials, such as ceramics, is usually described by the Weibull statistical distribution, either two- or three-parametric, or the one corresponding to more fracture modes.[1–6] The reliability of the Weibull distribution has been theoretically and experimentally investigated for a very broad range of conditions.[7–23] One of the typical experimental problems is that the cost limits the number of testing pieces for the strength measurements, which makes the prediction of the free parameters in the chosen distribution less reliable.

Different calculation procedures are used to evaluate the Weibull parameters (or the corresponding parameters in other statistical distributions), the most popular being the linear-regression (LR) method and the maximum-likelihood (ML) method. Each of these methods has its benefits and drawbacks.[7,8] Monte–Carlo simulations are a very useful tool for predicting the reliability of various estimation methods and their optimization, particularly when they are combined with experiments. These simulations indicate that all of the estimation methods, such as LR and ML, show some biasing in the estimated parameters, depending on the size of the test group and the adaptation and optimization of the particular method. The maximum-likelihood method is a standard method due to its efficiency and its ease of application when censored failure populations are encountered.[24] Since this method has proved to be particularly suitable, several variants of it have been proposed and tested, for instance, the generalized maximum-likelihood method (GMLE), which uses various rank estimators.[25–27] In addition, some authors tested the idea of dividing several measured strength values of ceramic materials into random, smaller subsets in order to study the corresponding statistical distribution of the Weibull parameters.[4,5,8–10]

However, the justification for the use of the Weibull distribution has been addressed by many authors and several other distributions have been proposed, including the normal (Gaussian), log-normal and Gamma distributions.[2,28–31] The Weibull distribution cannot be favored with certainty as compared, for instance, with the Gaussian distribution, when a limited number of samples are subjected to the strength test.[29] The distribution may be changed, for instance, in non-homogeneous materials, such as composites and porous ceramics, due to the different mechanisms, e.g., residual stresses.

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When the amount of available experimental strength data is modest, it is usually impossible to state with certainty that the Weibull distribution is correct, and not, for instance, the Gaussian. Different methods offer some reliability factors, which enable a quantitative comparison of the successfulness when using different distributions for the same set of experimental data. An example is the correlation coefficient in the linear regression method, which measures the deviations of the data points from a straight line in the appropriate linearized dependence of the probability on the strength.

In statistics, $P-P$ plots (P standing for the cumulative probability) or $Q-Q$ plots (Q standing for the quantile) are often used to obtain a visual impression of how well the known theoretical distribution fits the experimental data.[32,33] A good match of plot points to the 45° line indicates a good agreement between the experiment and the theory for both types of plots (the physical units on both axes correspond to the probabilities in the case of the *P–P* plots, or the measured quantity in the case of the *Q–Q* plots).

In our previous paper we analyzed a large quantity of Monte–Carlo data and estimated the Weibull parameters using the ML method.[34] We combined theoretical results with the results of our measurements of the four-point bend strength of 96% alumina samples from a serial production (1000 strength values). We focused mostly on the problem of the reliability of the estimation of the Weibull modulus for a small number of samples. In particular, we confirmed the log-normal distribution of the estimated values of the Weibull modulus when a large set of data is randomly divided into small subsets.

In this work we make a statistical evaluation of 5100 experimental values of the bend strength of test pieces from a serial production of alumina products. We use the ML method for an estimation of the statistical parameters together with a *Q–Q*-plot to show that the Weibull distribution best fits the experimental data.

2. Experimental

Ceramic samples were fabricated using the low-pressure injection-molding technique in the company Hidria AET d.o.o. for quality-control purposes. The strengths of 5100 samples in the shape of a rectangular bar with dimensions of 4 mm × 3 mm × 45 mm, collected from 425 batches, were used in the study (there were 12 broken test pieces in each batch). The material was high alumina ceramic with a density of 0.95 of the theoretical value. The ceramic was prepared by sintering for 3 h at 1640 °C. The feedstock for injection molding was made from a powder containing 96% alumina (d_{10} = 0.7 μm, d_{50} = 1.9 μm, d_{10} = 4.2 μm) and 4% silica-based material (d_{10} = 0.7 μm, d_{50} = 4.8 μm, d_{90} = 9.5 μm), which served as a liquid-phase sintering aid. The numbers in brackets correspond to the particle diameters, where the cumulative size distribution reaches values of 10%, 50% and 90%, respectively. The material is primarily used for electrical insulating purposes and not high-strength-demanding tasks and is labeled as a "middle-strength" alumina ceramic in the company.

The strength was calculated from the breaking force in a 4-point bending test[35] using the equation:

$$\sigma = \frac{3F(L_S - L_L)}{2ah^2} \tag{1}$$

where σ is the bending strength, F is the breaking force, L_S = 40 mm is the outer support span, L_L = 20 mm is the load span, a = 4 mm is the specimen width, and h = 3 mm is the specimen thickness.

3. Statistical model and graphical representation

Our statistical variable is the four-point bend strength (called strength for brevity), denoted by the symbol σ. In our calculations we deal with both probability distribution functions: the probability density function $p(\sigma)$, and the cumulative probability function, also called the unreliability function, which is defined as: $P(\sigma) = \int_0^\sigma p(x) \quad dx$. We test the statistical compatibility of the experimental data with the four different 2-parametric distribution functions: (1) Weibull, (2) normal (Gaussian), (3) log-normal, and (4) Gamma. The exact mathematical formulae are described in Section 3.2.

3.1. The procedure to estimate the goodness of fit

The goodness of fit for a specific distribution was estimated from probability plots, where the experimental data is plotted against values calculated with a theoretical distribution. This is a graphical technique for assessing how well a certain distribution can describe experimental data. The graphical method, where all the experimental data are plotted, gives an important qualitative estimation about how well a particular distribution describes the data. The visualization of all the strength data in the evaluation of the distribution reliability is more illustrative and trustworthy than merely giving a number that indicates the level of correspondence of the theoretical distribution to the real experimental data.

The detailed procedure for constructing probability plots for each considered statistical distribution consists of the following steps:

(a) The best fitting parameters are determined for each distribution by using the maximum-likelihood method. This is done in the following way. The (N = 5100) measured strength values, σ_i, i = 1 to N, are inserted into the probability density function $p(a,b;\sigma)$, where a and b stand for the corresponding free parameters of the distribution, e.g., $a \equiv m$ and $b \equiv \sigma_0$ for the Weibull distribution, etc. The ML procedure maximizes the following function with respect to the free parameters a and b:

$$Y = \ln\left(\prod_{i=1}^{N} p(a,b;\sigma_i)\right) = \sum_{i=1}^{N} \ln \quad p(a,b;\sigma_i) \tag{2}$$

by setting to zero the derivatives of Y with respect to a and b. The detailed procedure is different for each distribution. The equations that were used to calculate each parameter for a particular distribution are described in the Mathematical appendix.

(b) The experimental bending-strength values σ_i are ranked in an ascending order, where i is the consecutive number of the sample in the ordered set. A probability of failure (P_i) is assigned to each value, according to a standard estimator[24]:

$$P_i = \frac{i - 0.5}{N} \tag{3}$$

where P_i is the cumulative probability of failure, and N is the total number of samples (5100 in our case). Other estimators (see comments in Section 3) were also checked; it was revealed that the choice of estimator function has no effect on the results, so that the use of Eq. (3) is justified.

(c) For each P_i a corresponding theoretically expected strength $\sigma_{i,\text{th}}$ is calculated from the inverse cumulative distribution, using the parameters determined with the maximum-likelihood method. This was done either analytically or numerically by inverting the corresponding equations for the cumulative distribution functions in Section 3.2.

(d) We use the graphical representation of the theoretical vs. experimental distribution in a manner similar to the construction of the *Q–Q* probability plots. On our "*Q–Q*" plots the i,th experimental value of the strength σ_i corresponds to the coordinate on the vertical axis and the theoretically expected strength $\sigma_{i,\text{th}}$ corresponds to the coordinate on the horizontal axis. If the experimental data are in excellent agreement with the proposed theoretical distribution, then all the points on the graph lie very near the straight line with an inclination of 45° ($y = x$ line, for briefness).

(e) A correlation coefficient or R^2 factor is calculated to determine how close to the $y = x$ line on average the plot points lie:

$$R^2 = 1 - \frac{\sum_{i=1}^{N}(\sigma_i - \sigma_{i,\text{th}})^2}{\sum_{i=1}^{N}(\sigma_i - \langle\sigma_i\rangle)^2} \tag{4}$$

where $\langle\sigma_i\rangle$ is the mean value of the experimental strengths.

3.2. Distributions

One of the two parameters (the second one) for all the four distributions will be denoted similarly: $\sigma_{0\text{W}}$, $\sigma_{0\text{N}}$, $\sigma_{0\text{LN}}$ or $\sigma_{0\text{G}}$ (see below) to stress the similarity of its meaning. In all cases it has the dimension of strength and is directly proportional to the mean strength of the corresponding strength distribution. It can be called the scale parameter. In all the distributions below, the strengths are theoretically limited to non-negative values; only the normal distribution is formally widened to negative strengths (this has no practical consequence since physically senseless negative strengths are extremely improbable).

In order to be brief, we give this section only the probability functions for the four distributions, while in the Mathematical appendix the reader can find the corresponding formulae for the estimation of the parameters with the ML method, as well as some resulting statistical parameters (the mean value, standard and cubic deviations, see below).

(a) Weibull distribution

In the case of the 2-parameter Weibull statistics, the functions $p(\sigma)$ and $P(\sigma)$ are equal to:

$$p(\sigma) = \frac{m}{\sigma_{0\text{W}}}\left(\frac{\sigma}{\sigma_{0\text{W}}}\right)^{m-1} \cdot \exp\left(-\left(\frac{\sigma}{\sigma_{0\text{W}}}\right)^{m}\right) \tag{5a}$$

$$P(\sigma) = 1 - \exp\left(-\left(\frac{\sigma}{\sigma_{0\text{W}}}\right)^{m}\right), \tag{5b}$$

with the Weibull modulus m and the scale parameter $\sigma_{0\text{W}}$.

(b) Normal distribution

The p and P functions for the normal distribution are:

$$p(\sigma) = \frac{1}{\delta \cdot \sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}\left(\frac{\sigma - \sigma_{0\text{N}}}{\delta}\right)^2\right), \tag{6a}$$

$$P(\sigma) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{\sigma - \sigma_{0\text{N}}}{\delta}\right)\right), \tag{6b}$$

where the parameters have the simple and direct meaning: $\sigma_{0\text{N}}$ is the mean strength, and δ is the standard deviation. The tabulated error function $\text{erf}(x) = \sqrt{2/\pi}\int_0^x \exp(-t^2/2)dt$ is used for the cumulative probability function.

(c) Log-normal distribution

A log-normal distribution means that the logarithms of the strengths are distributed normally. The corresponding p and P functions are:

$$p(\sigma) = \frac{1}{\sigma} \cdot \frac{1}{w \cdot \sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}\left(\frac{\ln\sigma - \ln\sigma_{0\text{LN}}}{w}\right)^2\right), \tag{7a}$$

$$P(\sigma) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{\ln\,\sigma - \ln\,\sigma_{0\text{LN}}}{w}\right)\right), \tag{7b}$$

Note the additional factor $1/\sigma$ in expression (7a) for $p(\sigma)$, as compared to $p(\sigma)$ for the normal distribution in Eq. (6a). Here, the parameter w is dimensionless, while the second parameter, $\sigma_{0\text{LN}}$, is written with logarithms so that the physical units match.

(d) Gamma distribution

In the case of the Gamma distribution, the functions $p(\sigma)$ and $P(\sigma)$ are equal to:

$$p(\sigma) = \frac{1}{\sigma_{0\text{G}}^k \cdot \Gamma(k)} \cdot \sigma^{k-1} \cdot \exp\left(-\frac{\sigma}{\sigma_{0\text{G}}}\right) \tag{8a}$$

$$P(\sigma) = \frac{1}{\Gamma(k)} \cdot \int_0^{\sigma/\sigma_{0G}} t^{k-1} \cdot \exp(-t)dt, \tag{8b}$$

with the dimensionless parameter k and the scale parameter $\sigma_{0\text{G}}$. Note that P has no analytical form and must be tabulated. Here, $\Gamma(k) = \int_0^\infty t^{k-1}e^{-t}dt$ is the gamma function.

4. Results and discussion

All the strength measurements are presented in Fig. 1, where each value is plotted in chronological order. The experimental

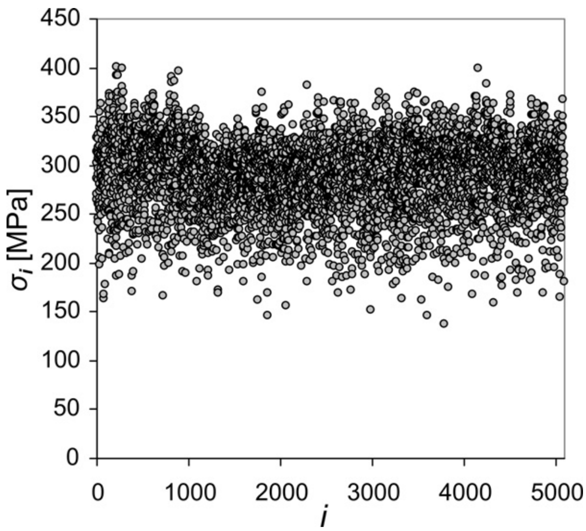

Fig. 1. All 5100 bending strength values with the mean value of 289.557 MPa (unsorted strength data); the numbers on the horizontal axis correspond to ordinal numbers of the data.

Table 1
ML parameters and the R^2 factor for the four distributions.

Distribution	1st parameter	2nd parameter	R^2
Weibull	$m = 9.048$	$\sigma_{0W} = 305.54$ MPa	0.9984
Normal	$\delta = 37.486$ MPa	$\sigma_{0N} = 289.557$ MPa	0.9855
Log-normal	$w = 0.1372$	$\sigma_{0LN} = 286.86$ MPa	0.9468
Gamma	$k = 55.599$	$\sigma_{0G} = 5.208$ MPa	0.9645

mean value of the strengths calculated from the arithmetic average of the 5100 strength values is $\langle\sigma\rangle = 289.557$ MPa.

Table 1 gives the ML-estimated parameters for all four theoretical distributions, together with the R^2 factor, and the corresponding probability plots for the different distributions are presented in Figs. 2–5. The first parameter (the shape parameter) is dimensionless, except for the normal distribution. The second parameter (the scale parameter) has the dimensions of strength.

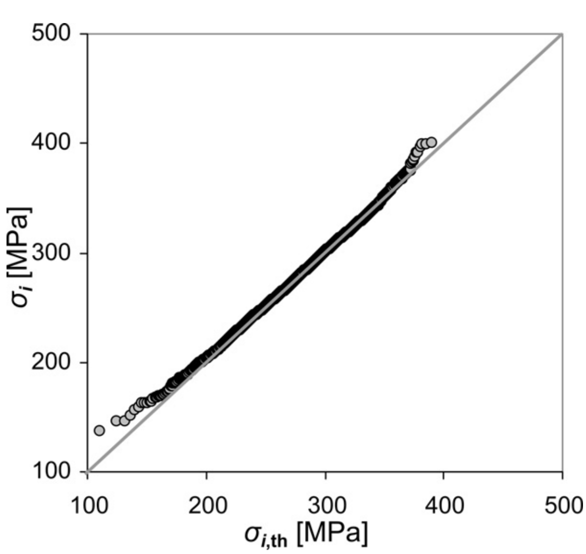

Fig. 2. Probability plot for the Weibull distribution.

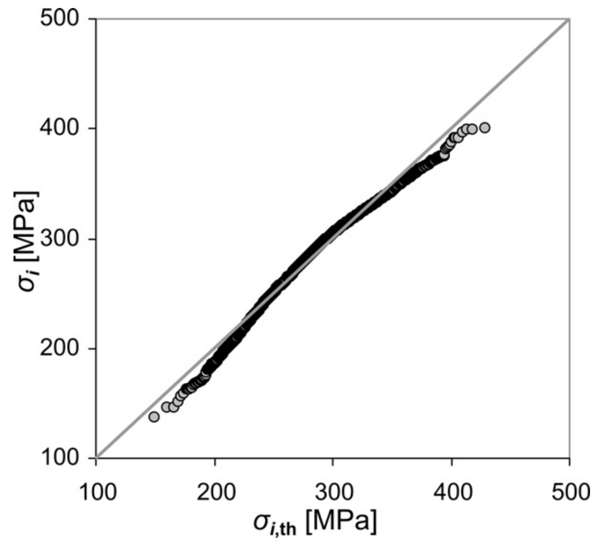

Fig. 3. Probability plot for the normal distribution.

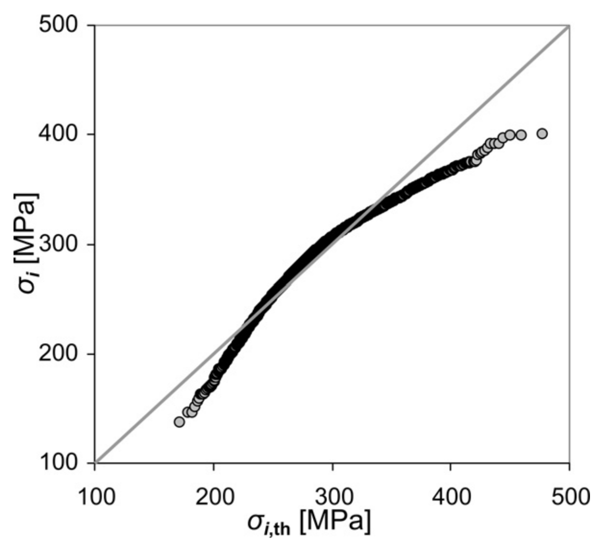

Fig. 4. Probability plot for the log-normal distribution.

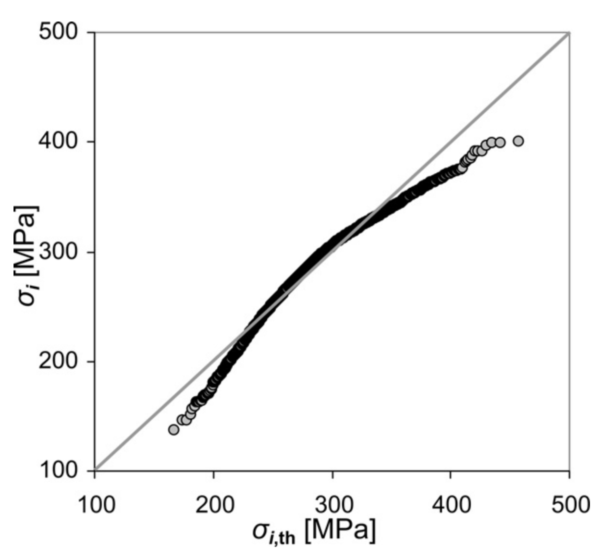

Fig. 5. Probability plot for the gamma distribution.

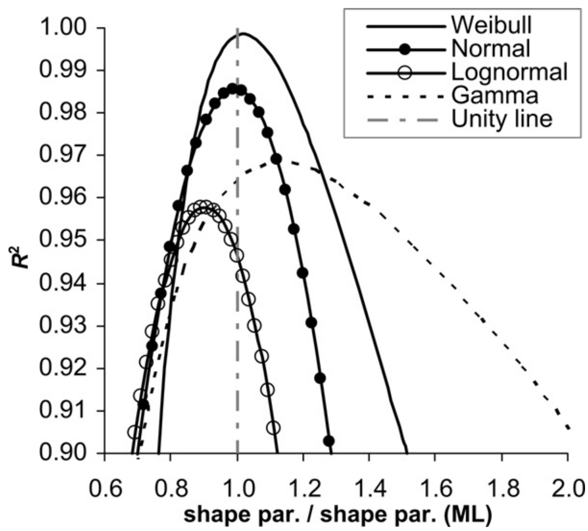

Fig. 6. Dependence of the R^2 factor on the shape parameter if the theoretical mean strength is kept equal to the experimental average. The shape parameters are m, δ, w and k, respectively, for the four distributions. The shape parameter is normalized with respect to the value obtained with the ML method.

From a visual inspection the data points fit best to a straight line in the case of the Weibull distribution, and this is confirmed by the highest R^2 factor. The normal distribution is not significantly worse than the Weibull distribution, but the remaining two distributions are definitely less successful in describing the strength data. The plot points in all the graphs are much denser in the middle than for the small and large strength values, which is already evident in Fig. 1.

The influence of choosing different rank estimators was verified by taking some other commonly used expressions[7,13,17,11,36] instead of Eq. (3), for instance:

$$P_i = \frac{i - 0.3}{N + 0.4} \tag{9}$$

The R^2 factors varied very little. The largest discrepancy appeared in the log-normal distribution, where the difference between the R^2 factors was only 0.06%.

However, we must be aware that the parameters optimized using the maximum-likelihood method do not necessarily yield the highest possible R^2 factor for the given distribution. In order to clarify this question, we optimized the R^2 factor for each distribution by varying the first (shape) parameter in Table 1 around the value obtained using the ML method. The second parameter was adjusted to the first to keep the theoretical mean strength value (see Mathematical appendix) the same as for the experimental average strength (the second parameter in the normal distribution was kept constant, of course, since it coincides with the average strength). The dependence of R^2 on the shape parameter was plotted in Fig. 6; this parameter was scaled relative to its ML value (see the vertical line in the figure). The maximum of R^2 corresponds exactly to the ML value of the scale parameter only for the normal distribution; this could be expected since the ML procedure seems to work optimally for this distribution. In the case of the Weibull distribution the maximum is only slightly shifted with respect to the value with the ML parameters; the discrepancy for the Gamma and log-normal distributions is somewhat larger. Nevertheless, it is evident, first that the ML method gives an almost optimal R^2 factor for all distributions, and second, that the Weibull distribution truly describes the experimental data best.

Finally, we make a comparison for the four distributions to see how well the theoretical mean strength $\langle\sigma\rangle$, its standard deviation $\delta\sigma$ and the cubic deviation $\delta_3\sigma$ fit to the experimental values. The cubic deviation can be defined in a similar manner as the standard deviation:

$$\delta_3\sigma = \sqrt[3]{\langle(\sigma - \langle\sigma\rangle)^3\rangle} \tag{10}$$

The brackets $\langle\ldots\rangle$ denote statistical averaging. The usefulness of this deviation is that it reveals the asymmetry of the distribution around its maximum. The results are shown in Table 2. In the first row there are the experimental values of these statistical parameters, followed by the theoretically obtained values for the four distributions. The values for each of the three parameters are presented in two columns: in the left column there is a value directly calculated from the ML parameters (see Mathematical appendix), while in the right column the parameters are obtained from the simulated "theoretical" strengths in the probability plots, using the corresponding statistical averaging.

Table 2
Experimental and calculated statistical parameters for the four distributions.

	$\langle\sigma\rangle$		$\delta\sigma$		$\delta_3\sigma$	
Experiment	289.56		37.49		−29.58	
Weibull	289.41	289.41	38.26	38.26	−32.15	−32.12
Normal	289.56	289.56	37.49	37.48	0	1.19
Log-normal	289.57	289.56	39.92	39.91	29.80	29.75
Gamma	289.56	289.56	38.83	38.83	25.04	25.01

While the mean value and the standard deviation of the calculated strength for all the distributions agree reasonably well with the experimental values (the exact agreement holds for the normal distribution), only the Weibull distribution gives a nearly correct result for the cubic deviation.

5. Conclusion

The comparison of four different theoretical statistical distributions with 5100 experimental strengths for liquid-phase-sintered alumina revealed that the Weibull distribution most accurately describes the strength scatter. The R^2 factor obtained by using the parameters from the maximum-likelihood method and by comparing theoretically simulated strengths with experimental values was the highest for the Weibull distribution: $R^2 = 99.84\%$. In addition, the mean cubic deviation of the strength values from the mean strengths in the experimental data is not negligible (it is comparable to the standard deviation) and is correctly estimated only for the Weibull distribution.

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We would gladly provide the full set of the experimental data in the digital form to the interested reader (contact the corresponding author).

Appendix A. Mathematical appendix

(a) Weibull distribution

When the derivatives of Y in Eq. (1) for the ML method with respect to m and σ_0 are set to zero, the following pair of equations is obtained:

$$\frac{1}{m} = \frac{\sum_{i=1}^{N} \ln \sigma_i \cdot {\sigma_i}^m}{\sum_{i=1}^{N} \sigma_i^m} - \frac{1}{N}\sum_{i=1}^{N} \ln\, \sigma_i$$

$$\sigma_{0W} = \left(\frac{\sum_{i=1}^{N} \sigma_i^m}{N}\right)^{1/m}$$

In practice, the first equation is first numerically solved for m and then $\sigma_{0\mathrm{W}}$ is calculated from the second equation.

The theoretical mean value of the strength $\langle\sigma\rangle$, its standard deviation $\delta\sigma$ and cubic deviation $\delta_3\sigma$ are equal to:

$$\langle\sigma\rangle = \sigma_{0\mathrm{W}} \cdot \Gamma\left(1 + \frac{1}{m}\right)$$

$$\delta\sigma = \sigma_{0\mathrm{W}} \cdot \sqrt{\Gamma\left(1+\frac{2}{m}\right) - \Gamma^2\left(1+\frac{1}{m}\right)}$$

$$\delta_3\sigma = \sigma_{0\mathrm{W}} \cdot \sqrt[3]{\Gamma\left(1+\frac{3}{m}\right) + 2\Gamma^3\left(1+\frac{1}{m}\right) - 3\Gamma\left(1+\frac{1}{m}\right)\Gamma\left(1+\frac{2}{m}\right)}$$

where $\Gamma = \int_0^\infty t^{x-1} e^{-t} dt$ is the gamma function.

(b) Normal distribution

The ML method gives the same equations for the mean value and standard deviation of the strengths as is normally used in a statistical evaluation of the data without reference to the exact form of the distribution function:

$$\sigma_{0\mathrm{N}} = \langle\sigma\rangle \equiv \frac{1}{N}\sum_{i=1}^{N}\sigma_i$$

$$\delta = \delta\sigma \equiv \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\sigma_i - \sigma_{0N})^2}$$

while the cubic deviation is zero because of the symmetry of the Gaussian function.

(c) Log-normal distribution

The corresponding ML parameters are obtained in a similar manner as for the normal distribution:

$$\ln \sigma_{0\mathrm{LN}} = \frac{1}{N}\sum_{i=1}^{N}\ln \sigma_i$$

$$w = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\ln \sigma_i - \ln \sigma_{0\mathrm{LN}})^2}$$

The theoretical mean strength, standard and cubic deviations are:

$$\langle\sigma\rangle = \sigma_{0\mathrm{LN}} \quad \exp\left(\frac{w^2}{2}\right)$$

$$\delta\sigma = \sigma_{0\mathrm{LN}} \quad \exp\left(\frac{w^2}{2}\right) \cdot \sqrt{\exp(w^2) - 1}$$

$$\delta_3\sigma = \sigma_{0\mathrm{LN}} \quad \exp\left(\frac{w^2}{2}\right) \cdot \sqrt[3]{\exp(3w^2) - 3\exp(w^2) + 2}$$

(d) Gamma distribution

The ML equations for the parameters are:

$$\frac{d\Gamma(k)/dk}{\Gamma(k)} - \ln k = \frac{1}{N}\sum_{i=1}^{N}\ln \sigma_i - \ln\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i\right) \equiv \langle\ln \sigma\rangle - \ln\langle\sigma\rangle$$

$$\sigma_{0\mathrm{G}} = \frac{\sum_{i=1}^{N}\sigma_i}{Nk} \equiv \frac{\langle\sigma\rangle}{k}$$

In practice, the first equation is first numerically solved for k and then $\sigma_{0\mathrm{G}}$ is calculated from the simple second equation. This strategy is comparable to that of solving the similar ML problem for the Weibull distribution.

The theoretical mean strength, standard and cubic deviations are:

$$\langle\sigma\rangle = k\sigma_{0\mathrm{G}}$$

$$\delta\sigma = \sqrt{k} \cdot \sigma_{0\mathrm{G}}$$

$$\delta_3\sigma = \sqrt[3]{2k} \cdot \sigma_{0\mathrm{G}}$$

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